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On j-Convex Preserving Interpolation Operators

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We present results regarding the existence of j-convex preserving interpolation operators, as well as results concerning the determination of existence of such operators. We include an application in which we make use of a sufficient set of test-functions to characterize when every degree of convexity can be preserved among particular families of polynomial interpolation operators, which include the Bernstein operators. © 2000 Academic Press

1. INTRODUCTION

Let $(X, \|\cdot\|) = (C[a, b], \|\cdot\|_\infty)$ where $\|f\|_\infty = \sup_{t \in [a, b]} |f(t)|$. In [12, p. 26], the notion of an *interpolation operator* is introduced: a linear operator $P\colon X \to X$ that can be written $P = \sum_{i=0}^{m} \delta_{t_i} \otimes v_i$, where $t_i \in [a, b]$, $t_i < t_{i+1}$ and $v_i \in X$, $i = 0, ..., m$, is said to be an interpolation operator; note that in this terminology we do not mean that Pf interpolates f at the point t_i, but rather that Pf is determined by the evaluation of f at each t_i. As such, we may assume that $t_0 = a$ and $t_m = b$. We denote the set of interpolation operators by $\mathscr{P}$. In the literature, these operators are also referred to as *finite carrier* or *discretely defined* operators.

Interpolation operators have been the topic of recent study, particularly with regard to establishing Jackson-type estimates in the approximation of functions from $C[a, b]$. Starting from [2], for example, variations on the following question have been addressed (see, e.g., [4, 17]): For each $n \in \mathbb{N}$, can one construct a positive-preserving interpolation operator, L_n, such that

$$\|L_n f - f\|_{C[a, b]} = \mathcal{O}(n^{-\alpha})$$

for all f such that $\omega_2(f, \delta) \leqslant C\delta^\alpha$, $0 \leqslant \alpha \leqslant 2$? Pointwise estimates to the above problem have also been considered (see [13, 14]). As a further variation, pointwise estimates have been given for operators (not necessarily interpolatory) that preserve higher degrees of convexity (see, e.g., [5, 14]).

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It is in [14] that convexity-preserving results pertaining to certain interpolation operators are given; the authors exhibit a theorem which characterizes, via a finite set of "test-functions," the convexity-preserving ability of certain interpolation operators. The main results of the current paper use a finite set of test-functions in an attempt to generalize this characterization. We determine a finite test-function set that provides a sufficient condition for the preservation of arbitrary j-convexity; this set also offers a characterization of j-convex preservation in particular situations. We go on to show in Section 3 that this characterization is best possible, in the sense that no finite set of functions can characterize j-convex preservation in any other situation. In Section 4, motivated by [3, 8], the sufficient conditions given in Section 2 are utilized in characterizing when it is possible for *Bernstein-type* operators to preserve all degrees of convexity.

The set of j-convex functions of X, which we now denote as S, may be defined in a number of equivalent manners (e.g., see [1, 15]). However, due to the approach of the current considerations, it is most convenient for us to define S via the following mechanism. For a nonnegative integer j, we denote the jth divided difference of $f \in X$ at the points $a \leqslant x_0 < \cdots < x_j \leqslant b$ by $[x_0, x_1, ..., x_j] f$. Of course $[x_0, x_1, ..., x_j] f$ can be expressed as a linear combination of point evaluations of $f(x)$ and thus we regard $[x_0, ..., x_j] \in X^*$ where $\langle f, [x_0, ..., x_j] \rangle := [x_0, x_1, ..., x_j] f$. For a collection of functions $f_1, ..., f_n$, we denote the set of all nonnegative linear combinations of those functions by $cone(f_1, ..., f_n)$.

DEFINITION 1.1. For fixed j, let $S^* \subset X^*$ denote the weak*-closure of the cone generated by the set $S_0^* = \{[x_0, ..., x_j] \mid a \leqslant x_0 < \cdots < x_j \leqslant b\}$. Let $S = \{f \in X \mid \langle f, \phi \rangle \geqslant 0 \ \forall \phi \in S^*\}$. $f \in X$ is said to be j-convex if $f \in S$.

DEFINITION 1.2. $P \in \mathscr{P}$ is said to be j-convex preserving if Pf is j-convex whenever f is j-convex (i.e., $PS \subset S$).

Note 1. It follows (e.g., from Lemma 1.1 in [7]) that S^* defined above is exactly the *dual cone* of S; that is, $S^* \subset X^*$ is the set of all functionals nonnegative against S. Thus we have for any linear operator P, $PS \subset S$ if and only if $P^*S^* \subset S^*$ where P^* denotes the adjoint of P. The following theorem from [15] characterizes S^* by identifying the extreme rays of S (modulo Π_{j-1}).

THEOREM 1.1 (see [15, p. 407]). *Let S denote the cone of j-convex functions and let $S^* \subset X^*$ denote the dual cone of S. Then $u \in S^*$ if and only if*

$$\langle x^i, u \rangle = 0, \qquad i = 0, ..., j-1$$

and, for each $t \in [a, b]$,

$$\langle \phi_t(x), u \rangle \geqslant 0,$$

where

$$\phi_t = \begin{cases} 0 & \text{if} \quad x \leqslant t \\ (x-t)^{j-1} & \text{if} \quad x > t. \end{cases}$$

Note 2. Note 1 indicates that $P = \sum_{i=0}^{m} \delta_{t_i} \otimes v_i$ preserves j-convexity if and only if the linear combination of point-evaluations $P^*[x_0, ..., x_j]$ is a nonnegative linear combination of jth divided differences. The fragile nature of this problem is well illustrated in the following example, where we note that "small" changes in an operator's action can produce "large" consequences with respect to shape-preservation.

EXAMPLE 1.1. The second degree Bernstein operator $B_2 = C[0, 1] \to \Pi_2$ is an interpolation operator that preserves (among other things) 1-convexity or monotonicity. This is accomplished while nearly fixing Π_2; with respect to the basis $(1, t, t^2)^T$, the so-called *action matrix* associated with B_2 is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}.$$

However, as is well known (see, for example, [16]), an interpolation operator with identity action matrix (i.e., an operator that fixes Π_2) cannot preserve monotonicity. Indeed, employing the language of Notes 1 and 2 above, if such an operator $P: C[0, 1] \to \Pi_2$ did exist, we could rewrite it as $P = \sum_{i=1}^{3} u_i \otimes t^{i-1}$ where each u_i is a linear combination of point-evaluations. Then P would also preserve monotonicity from $(C^1[0, 1], \|\cdot\|)$ onto Π_2, where $\|f\| = \max_{i=0,1}\{\|f^{(i)}\|_\infty\}$. But this is in contradiction to [6, Lemma 2,2] which shows that such an operator, $P = \sum_{i=1}^{3} u_i \otimes t^{i-1} : C^1 \to \Pi_2$, must have $u_2 = \delta_0'$, where δ_0' denotes derivative evaluation at $t = 0$.

2. A SUFFICIENT (AND OCCASIONALLY NECESSARY) FINITE TEST-FUNCTION SET

As described in the following definition, we are interested in the possibility of subsets of X to which one may confine one's attention when determining if $P \in \mathscr{P}$ preserves j-convexity. We will thus assume in the following that

$P\Pi_{j-1} \subset \Pi_{j-1}$, since this condition is necessary in order for P to preserve j-convexity (note $P(\pm x^i)$ must be j-convex for $i=0, ..., j-1$). We also assume the necessary condition that $m \geqslant j$.

DEFINITION 2.1. Fix $a=t_0<t_1< \cdots <t_m=b$ and let $\Omega \subset X$. Let $\mathscr{P}_j$ denote the operators of $\mathscr{P}$ that leave Π_{j-1} invariant. We say Ω is a sufficient test-function set if, for all $P \in \mathscr{P}_j$, $P\Omega \subset S$ implies P preserves j-convexity. Similarly, we say Ω is a necessary test-function set if $P\Omega \subset S$ whenever $P \in \mathscr{P}_j$ preserves j-convexity.

We now construct a finite subset of X that is a sufficient test-function set. For particular choices of j and m, we find this set to be a necessary test-function set as well. It is interesting to note that the test-functions given below were also utilized in [11] for a different purpose.

DEFINITION 2.2. For the integer $j \geqslant 2$ and for $k=0, ..., m-j$, define

$$\omega_k^+(x) := \begin{cases} 0 & \text{for} \quad a \leqslant x \leqslant t_{k+j-1} \\ \omega_k(x) & \text{for} \quad t_{k+j-1} \leqslant x \leqslant b, \end{cases}$$

where $\omega_k(x) := (x-t_{k+1}) \cdots (x-t_{k+j-1})$. For $j=1$, define $w_k^+(x)$ to be the continuous piecewise linear function vanishing on $[a, (t_k+t_{k+1})/2]$, identically 1 on $[t_{k+1}, b]$ and linear on $[(t_k+t_{k+1})/2, t_{k+1}]$ (rising from 0 to 1 on this interval). Let $\Omega = \{\omega_k^+\}_{k=0}^{m-j}$.

Note 3. Since elements of $\mathscr{P}$ depend only on the points $t_0, ..., t_m$, the above set Ω, relative to elements of $\mathscr{P}$, is not unique; any set of $m-j+1$ functions agreeing with each ω_k^+ at the t_i's would suffice.

LEMMA 2.1. *Let $P=\sum_{i=0}^m \delta_{t_i} \otimes v_i$ be an interpolation operator and let $f \in S$. Then there exists $\omega_f \in cone(\omega_0^+, ..., \omega_{m-j}^+)$ and $q_f \in \Pi_{j-1}$ such that $Pf = P\omega_f + Pq_f$.*

Proof. Let $f(x) \in S$ and, for $k=0, ..., m-j$, let $p_k(f; x)$ be the $j-1$-degree polynomial that interpolates $f(x)$ at the j points $t_k, t_{k+1}, ..., t_{k+j-1}$. Note that $p_{k+1} - p_k$ has $(j-1)$ zeros and can be used to define $\omega_k(x)$,

$$\begin{aligned} p_{k+1}(f; x) - p_k(f; x) &= B_k(x-t_{k+1})(x-t_{k+2}) \cdots (x-t_{k+j-1}) \\ &= B_k w_k(x) \end{aligned}$$

for some constant B_k (in the $j=1$ case, we define $\omega_k(x) \equiv 1$). Since the remainder for polynomial interpolation can be expressed as

$$f(x) - p_k(f; x) = \langle f, [x, t_k, ..., t_{k+j-1}] \rangle (x-t_k) \cdots (x-t_{k+j-1})$$

we use the observation that $f(t_{k+j}) - p_k(f; t_{k+j}) - B_k w_k(t_{k+j}) = 0$ to solve for B_k in terms of a divided difference,

$$B_k = \langle f, [t_k, ..., t_{k+j}] \rangle (t_{k+j-t_k}). \tag{1}$$

Note that each $B_k \geqslant 0$ since $f \in S$. Now, for $j \geqslant 1$, define $\omega_k^+(x)$ via $\omega_k(x)$ and simply note that $f(x)$ and $p_0(f; x) + \sum_{k=0}^{m-j} B_k w_k^+(x)$ agree at t_i, $i = 0, ..., m$. Hence $P_f = Pp_0(f; x) + P\omega_f$ where $\omega_f := \sum_{k=0}^{m-j} B_k w_k^+(x) \in$ *cone*$(\omega_0^+, ..., \omega_{m-j}^+)$. ∎

THEOREM 2.1. *Ω is a sufficient test-function set.*

Proof. Let $[x_0, ..., x_j] \in S^*$ and let $f \in S$. Suppose $P \in \mathscr{P}_j$ is such that $P\Omega \subset S$. Then, by Lemma 2.1,

$$\langle Pf, [x_0, ..., x_j] \rangle = \langle P\omega_f + Pq_f, [x_0, ..., x_j] \rangle = \langle P\omega_f, [x_0, ..., x_j] \rangle \geqslant 0$$

since $Pq_f \in \Pi_{j-1}$ and $\omega_f \in$ *cone*$(\omega_0^+, ..., \omega_{m-j}^+)$. Thus P preserves j-convexity. ∎

THEOREM 2.2. *If $j = 1, 2$ then Ω is a necessary and sufficient test-function set.*

Proof. In the $j = 1, 2$ cases, $\Omega \subset S$. ∎

THEOREM 2.3. *If $m = j$, $j + 1$ then Ω is a necessary and sufficient test-function set.*

Proof. The sufficiency of Ω follows from Lemma 2.1. Suppose P preserves j-convexity. Let $m = j + 1$; note that in this case, $\Omega = \{\omega_0^+, \omega_1^+\}$. By Theorem 1.1, $\phi_{t_1} \in S$; let us consider the associated function $\omega_{\phi_{t_1}} = B_0 \omega_0^+ + B_1 \omega_1^+$. From (1) in the proof of Lemma 2.1, we see that $B_1 = 0$. Then for any $[x_0, ..., x_j] \in S^*$, we have

$$\begin{aligned} \langle PB_0\omega_0^+, [x_0, ..., x_j] \rangle &= \langle P(B_0\omega_0^+ + B_1\omega_1^+), [x_0, ..., x_j] \rangle \\ &= \langle P\phi_{t_1}, [x_0, ..., x_j] \rangle \\ &\geqslant 0. \end{aligned}$$

Since $B_0 \geqslant 0$, we conclude that $P\omega_0^+ \in S$. Similarly, using $\phi_{t_{m-1}}$ and the associated function $\omega_{\phi_{t_{m-1}}}$ one finds $P\omega_1^+ \in S$. In the $m = j$ case, we have $\Omega = \{\omega_0^+\}$, with ω_0^+ identically 0 on $[a, t_{m-1}]$. And thus ω_0^+ and $\phi_{t_{m-1}}$ differ on $\{t_0, t_1, ..., t_m\}$ by only a positive scalar multiple. Hence it follows that $P\omega_0^+ \in S$. ∎

Note 4. The extreme rays ϕ_t of S described in Theorem 1.1 can be associated with their corresponding $\omega_{\phi_t} \in cone(\omega_0^+, ..., \omega_{m-j}^+)$. Indeed, for each $\phi_t(x)$ we have $\omega_{\phi_t} := \sum_{k=0}^{m-k} B_k(t)\, w_k^+(x)$ where $B_k(t)$ is (a positive scalar multiple of) the B-spline $\langle \phi_t(x), [t_k, ..., t_{k+j}] \rangle$. Thus if $P \in \mathscr{P}$ preserves j-convexity then

$$\sum_{k=0}^{m-j} B_k(t)\, P w_k^+(x)$$

is j-convex for $t \in [t_0, t_m]$.

Proposition 2.1. *Let $P \in \mathscr{P}_j$. Then P preserves j-convexity if and only if*

$$\sum_{k=0}^{m-j} B_k(t)\, P w_k^+ \in S \qquad \textit{for all} \quad t \in [t_0, t_m].$$

3. THE IMPOSSIBILITY OF A NECESSARY AND SUFFICIENT FINITE TEST-FUNCTION SET

In this section we show that the cases of Section 2 in which a finite test-function set characterized the preservation of j-convexity are the only such cases. Thus throughout the following we assume $m-1 > j > 2$.

Theorem 3.1. *Let $\Omega = \{\omega_1, ..., \omega_n\} \subset X$. Then Ω is not a necessary and sufficient test-function set.*

The following lemma will be used in the proof of the above.

Lemma 3.1. *The cone $\hat{S}_{|_T}$ has infinitely many extreme rays, where $\hat{S} := S + \Pi_{j-1} \subset X/\Pi_{j-1}$ and $T := [\delta_{t_0}, ..., \delta_{t_m}] \cap (\Pi_{j-1})^{\perp}$.*

Proof. Π_{j-1} is a closed subspace of X, and thus X/Π_{j-1} is a Banach space. For $f \in X$, let $\hat{f} := f + \Pi_{k-1} \in X/\Pi_{j-1}$. Since the dual space of X/Π_{j-1} is isometrically isomorphic to $(\Pi_{j-1})^{\perp} \subset X^*$, we may regard $X/\Pi_{j-1} \subset X^{**}$ (note that, via this identification, we have $\langle u, \hat{x} \rangle = \langle x, u \rangle$ each for $u \in X^*$). Let $T := [\delta_{t_0}, ..., \delta_{t_m}] \cap (\Pi_{j-1})^{\perp}$ and note that T is of dimension $m-j+1$. We now construct a particular basis for T consisting of jth divided differences: for $i = 0, ..., m-j$ let

$$\Sigma_i = [t_i, t_{i+1}, ..., t_{i+j}].$$

The set $\{\Sigma_i\}_{i=0}^{m-j} \subset [\delta_{t_0}, ..., \delta_{t_m}]$ is linearly independent and, since each Σ_i is a jth divided difference, we have $\Sigma_i \in (\Pi_{j-1})^{\perp}$. Thus $\{\Sigma_i\}_{i=0}^{m-j}$ is a basis for T. Obviously $\hat{S} \subset X/\Pi_{j-1}$ is a pointed cone and thus so is $\hat{S}|_T$. We may regard $\hat{S}_{|_T}$ as a cone in R^{m-j+1} by associating to each $\hat{f}_{|_T} \in \hat{S}_{|_T}$ the vector

$$(\langle f, \Sigma_0 \rangle, ..., \langle f, \Sigma_{m-j} \rangle).$$

In demonstrating that $\hat{S}_{|_T}$ has infinitely many extreme rays, we may confine our attention to a particular 3-dimensional subcone of $\hat{S}_{|_T}$ as we now show. Note that, by Theorem 1.1, each extreme ray of $\hat{S}_{|_T}$ must contain the vector $\hat{\phi}_{t|_T}$ for some $t \in (t_0, t_m)$; i.e., the cone generated by $\{\hat{\phi}_{t|_T}\}_{t \in [t_0, t_m]}$ is exactly $\hat{S}_{|_T}$. Consider the subcone $K \subset \hat{S}_{|_T}$ generated by

$$\{\hat{\phi}_{t|_T}\}_{t \in (t_{m-3}, t_{m-2})}.$$

By the definition of $\phi_t(x)$ (in Theorem 1.1), it follows that

$$\langle \phi_t, \Sigma_i \rangle = \begin{cases} 0 & \text{if} \quad t \leqslant t_i \\ c_t & \text{if} \quad t_i < t < t_{i+j} \\ 0 & \text{if} \quad t \geqslant t_{i+j}, \end{cases} \tag{2}$$

where $c_t > 0$. Thus, if $\hat{\phi}_{t|_T} \in K$ (i.e., if $t \in (t_{m-3}, t_{m-2})$) then

$$\hat{\phi}_{t|_T} = (0, 0, ..., 0, \langle \phi_t, \Sigma_{m-j-2} \rangle, \langle \phi_t, \Sigma_{m-j-1} \rangle, \langle \phi_t, \Sigma_{m-j} \rangle), \tag{3}$$

and k is a 3-dimensional subcone (note that $m-j \leqslant m-3$ by our initial assumption of this section). Furthermore, it follows from (2) and (3) that K is not contained in the cone generated by $\{\hat{\phi}_{t|_T}\}_{t \in [t_0, t_{m-3}]}$. We now claim that K has infinitely many extreme rays. Indeed, by [18, p. 123], the B-splines $\langle \phi_t, \Sigma_{m-j-i} \rangle$ $(i = 0, 1, 2)$ appearing in (3), are linearly independent on (t_{m-3}, t_{m-2}); in fact, on this interval, we have

$$\langle \phi_t, \Sigma_{m-j-i} \rangle = \sum_{k=0}^{2} a_{i,k} (t_{m-j-k} - t)^{j-1}$$

for some constants $a_{i,k}$. Thus, with

$$\mathbf{v} := (\langle \phi_t, \Sigma_{m-j-2} \rangle, \langle \phi_t, \Sigma_{m-j-1} \rangle, \langle \phi_t, \Sigma_{m-j} \rangle)$$

and

$$\mathbf{w} := ((t_{m-j-2} - t)^{j-1}, (t_{m-j-1} - t)^{j-1}, (t_{m-j} - t)^{j-1})$$

we have $\mathbf{v} = \mathbf{w}M$, where M is a nonsingular 3×3 matrix. Clearly the cone generated by $\mathbf{w}$ has infinitely many extreme rays and thus so does $\mathbf{w}M$. Hence K has infinitely many extreme rays and thus $\hat{S}_{|_T}$ must have infinitely many extreme rays. ∎

Proof of Theorem 3.1. Let E denote the Banach space $X/+\Pi_{j-1}$. Let $\Omega=\{\omega_1, ..., \omega_n\} \subset X$ and consider $\hat{\Omega}_{|_T}$, where $\hat{\Omega} \subset E$, and $T:=[\delta_{t_0}, ..., \delta_{t_m}] \cap (\Pi_{j-1})^{\perp}$. Suppose that $\hat{\Omega}_{|_T}$ is contained in the cone $\hat{S}_{|_T}$. We claim that Ω is not a sufficient test-function set. Indeed, by Lemma 3.1, the cone $K:=cone(\hat{\omega}_{1|_T}, ..., \hat{\omega}_{1|_T})$ cannot contain all of $\hat{S}_T$; let $\hat{\omega}_{|_T} \in \hat{S}_T \cap \tilde{K}$. Let $C:=co(\hat{\omega}_{1|_T}, ..., \hat{\omega}_{1|_T})$ and note that the subspace $[\hat{\omega}_{|_T}]$ does not intersect C. From the convexity and compactness of C it follows that there exists an entire closed hyperplane H containing $[\hat{\omega}_{|_T}]$ such that $H \cap C=\varnothing$ (see [9, p. 112]). Thus there is a continuous linear functional $h \in (E_{|_T})^*$ such that, after scaling, $\langle \hat{\omega}_{|_T}, h\rangle=0$ and $\min_{x \in C}\langle x, h\rangle=1$. Now $\dim(T)=\dim(E_{|_T})=\dim((E_{|_T})^*)$, and therefore, again using the identification of E^* with $(\Pi_{j-1})^{\perp}$, we choose as a basis for $(E_{|_T})^*$ a (fixed) basis of T. Hence we may regard h as a linear combination of the point-evaluations $\{\delta_{t_0}, ..., \delta_{t_m}\}$ that vanishes on Π_{j-1}. To complete the proof, we "shift slightly" the above hyperplane so that it strictly separates C from $[\hat{\omega}_{|_T}]$. Indeed, let $g \in T$ be such that $\langle \hat{\omega}_{|_T}, g\rangle=1$. If $\max_{x \in C}\langle x, g\rangle \leqslant 0$, then take $\tau:=h-g$ so that $\langle \hat{\omega}_{|_T}, \tau\rangle=-1$, while $\langle x, \tau\rangle \geqslant 0$ for all $x \in C$. Otherwise let $\max_{x \in C}\langle x, g\rangle=: 1/c>0$ and take $\tau:=h-cg$ so that for every $x \in C$, we have $\langle x, \tau\rangle \geqslant 1-\langle x, cg\rangle \geqslant 0$, and $\langle \hat{\omega}_{|_T}, \tau\rangle=\langle \hat{\omega}_{|_T}, -cg\rangle=-c<0$. Regarding $\tau \in T$ as a linear combination of point-evaluations, we define $P:=\tau \otimes v$ for a fixed $v \in S$ where $v \notin \Pi_{j-1}$. Note that $P \in \mathscr{P}_j$. Furthermore, observe that, while $P\omega_i \in S$ for $i=1, ..., n$, we have $P\omega \notin S$ (where $\omega \in S$ is a function such that $\hat{\omega} \in E$ restricts to $\hat{\omega}_{|_T}$ on T). Thus Ω is not sufficient.

We now consider the case in which $\hat{\Omega}_{|_T} \not\subset \hat{S}_{|_T}$. In this case we claim that Ω is not a necessary test-function set. The proof of this claim is similar to the above argument: let $C=\overline{co}(\{\hat{\phi}_{t_{|_T}}\}_{t \in (0,1]})$, where the functions ϕ_t are as in Theorem 1.1 and the functions $\hat{\phi}_t \in E$ are normalized so that $\|\hat{\phi}_{t_{|_T}}\|=1$ for $t \in (0, 1]$. If $\hat{\Omega}_{|_T} \not\subset \hat{S}_{|_T}$, then, without loss, we may assume $\hat{\omega}_{1|_T} \notin \hat{S}_{|_T}$ and whence the space $[\hat{\omega}_{1|_T}] \cap C=\varnothing$. Thus, as demonstrated above, there exists $f \in T$ such that $\langle \hat{\omega}_{1|_T}, f\rangle<0$ and $\langle \hat{x}_{|T}, f\rangle>0$ for all $\hat{x}_{|_T} \in C$. Let $v \in S$ and define $P:=f \otimes v$. Then P is a j-convex preserving interpolation operator such that $P\omega_1 \notin S$. Thus Ω is not necessary. ∎

4. AN APPLICATION: BERNSTEIN-TYPE OPERATORS

In this section we are interested in nth degree polynomial interpolation operators, $P\colon C[0, 1] \to \Pi_n$, that are supported on the equidistant nodes $\{i/n\}_{i=0}^{n}$ (i.e., $P=\sum_{i=0}^{n} \delta_{i/n} \otimes v_i$ where the $v_i \in \Pi_n$). We specialize our consideration a bit more as given in the following definition.

DEFINITION 4.1. An interpolation operator $P=\sum_{i=0}^{n}\delta_{t_i}\otimes v_i$ is said to be an nth degree Bernstein-type operator if $t_i=i/n$ and, in addition, each v_i contains a t^i term but contains no t^j term, $j<i$. Let $\mathscr{B}_n$ denote the set of all nth degree Bernstein-type operators.

In [3], clever results regarding uniform convergence of particular sequences operators of Bernstein-type are given. The operators considered in [3] were obtained by replacing the binomial coefficients in the nth-degree Bernstein operator with general ones satisfying a particular recursive relation; i.e., $A_n=\sum_{k=0}^{n}\delta_{k/n}\otimes\alpha_{n,k}x^k(1-x)^{n-k}$. However, in general, operators of this form cannot preserve j-convexity for any $j=1, ..., n$. This is easily seen via a parameter count: in order for A_n to preserve j-convexity, it is necessary that A_n be invariant on Π_j. As illustrated below, this requirement translates into $j(n-j+1)$ conditions on the $\alpha_{n,k}$ coefficients, $k=0, ..., n$; the resulting coefficients give only positive scalar multiples of the original nth degree Bernstein operator.

Thus we are thus motivated to seek Bernstein-type operators that preserve varying degrees of convexity. The following theorem characterizes those operators preserving j-convexity for all $j=0, 1, ..., n$.

THEOREM 4.1. *Let P be an nth degree Bernstein-type operator. Then P preserves (simultaneously) j-convexity for $j=0, 1, ..., n$ if and only if P preserves* 0-*convexity or positivity.*

EXAMPLE 4.1. Of course the nth degree Bernstein operator belongs to $\mathscr{B}_n$. In the course of the proof of Theorem 4.1, it will be shown that in fact $\mathscr{B}_n$ forms an n-parameter family of operators. In general, within this family there are many operators that preserve positivity and thus preserve convexity of every degree. For example, in the $n=2$ case the 1-parameter family given by

$$P_c=(\delta_0\otimes c-ct+t^2)+(\delta_{1/2}\otimes ct-2t^2)+(\delta_1\otimes t^2)$$

preserves positivity for all $c\geqslant 2$.

To prove the above theorem we will need the following technical lemma.

LEMMA 4.1. *Let j, k, and h be positive integers such that $1\leqslant j\leqslant k\leqslant h$. Then*

$$\sum_{i=k}^{h}(-1)^{h-i}\binom{h}{i}\prod_{m=1}^{j-1}(i-k+m)=(-1)^{h-k}(j-1)!\binom{h-j}{k-j}, \tag{4}$$

where, in the $j=1$ case, $\prod_{m=1}^{j-1}(i-k+m):=1$.

Proof. We prove (4) by induction on $j\geqslant 1$, where the case $j=1$ is again proved by induction, this time on k. ∎

Proof of Theorem 4.1. We must show that every positive operator in $\mathscr{B}_n$ preserves j-convexity, for $j = 1, ..., n$. To this end, we begin by noting that $\mathscr{B}_n$ forms an $(n+1)$-parameter family in the following way: for $P \in \mathscr{B}_n$, we can write

$$P = \sum_{i=0}^{n} \delta_{t_i} \otimes v_i = \vec{\delta} \otimes \vec{v}$$

$$= (\delta_0, \delta_{1/n}, ..., \delta_1) \otimes \begin{pmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,n} \\ 0 & a_{1,1} & \cdots & a_{1,n} \\ \vdots & 0 & a_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{n,n} \end{pmatrix} \begin{pmatrix} 1 \\ t \\ \vdots \\ \vdots \\ t^n \end{pmatrix}. \tag{5}$$

The conditions that $P(\Pi_j) \subset \Pi_j$, $j = 0, ..., n-1$, give rise to $\frac{n(n+1)}{2}$ equations (linear in the coefficients) involving columns 1 through n of the above matrix. Specifically, for fixed h, where $1 \leqslant h \leqslant n$, we must have

$$\langle t^j, (\delta_0, \delta_{1/n}, ..., \delta_1) \rangle \cdot \begin{pmatrix} a_{0,h} \\ \vdots \\ a_{h,h} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0 \tag{6}$$

for each $j = 0, ..., h-1$. These h equations allow us to express all coefficients in column h in terms one parameter. We choose as parameters the diagonal entries and find that

$$a_{i,h} = a_{h,h} (-1)^{h-i} \binom{h}{i}, \qquad i = 0, ..., h \tag{7}$$

solves (6) for every $0 \leqslant j \leqslant h-1$. Letting $a_i := a_{i,i}$, $i = 0, ..., n$, we can rewrite (5) as

$$P = (\delta_0, \delta_{1/n}, ..., \delta_1) \otimes \begin{pmatrix} a_0 & -a_1 & a_2 & \cdots & (-1)^n a_n \\ 0 & a_1 & -2a_2 & \cdots & (-1)^{n-1} n a_n \\ 0 & 0 & a_2 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & a_n \end{pmatrix} \begin{pmatrix} 1 \\ t \\ \vdots \\ \vdots \\ t^n \end{pmatrix}.$$

Referring back to (5), we see that $P \in \mathscr{B}_n$ is positive if and only if each v_i is a nonnegative polynomial. Thus we may assume that the result of "dotting" each row of the coefficient matrix in (8) with $(1, t, ..., t^n)^T$ produces a nonnegative polynomial in t.

We demonstrate that P preserves j-convexity, for $j = 1, ..., n$, by induction. Thus we begin by verifying that P preserves 1-convexity or monotonicity. Referring to Theorem 2.1, let $\omega_{1,k}^+$ be a continuous piecewise linear function, vanishing on $[0, (k/n) - (1/2n)]$ and identically 1 on $[(k/n), 1]$, $k = 1, ..., n$ (note that, to simplify notation, we enumerate the ω^+ functions in this application in a slightly different manner). Then P preserves monotonicity if and only if $P\omega_{1,k}^+$ is monotone for each k. Using the fact that

$$\langle \omega_{1,k}^+, (\delta_0, \delta_{1/n}, ..., \delta_1) \rangle = (0_1, 0_2, ..., 0_k, 1, ..., 1),$$

we see that every term of polynomial $Pw_{1,k}^+(t)$ has degree greater than or equal k. The coefficient for t^h, $k \leqslant h \leqslant n$, is a partial sum of entries in column h in the matrix of (8); i.e., denoting by $c_{k,h}$ the coefficient of t^h, one finds that $c_{k,h} = a_h \sum_{i=k}^{h} (-1)^{h-i} \binom{h}{i}$. Using the $j = 1$ case of Lemma 4.1, one then has

$$c_{k,h} = a_k \frac{k}{h} (-1)^{h-k} \binom{h}{k}.$$

Now $(Pw_{1,k}^+)'(t) = \sum_{h=k}^{n} hc_{k,h} t^{h-1}$ and thus the coefficient of t^h in the polynomial $t/k(Pw_{1,k}^+)'(t)$ is $a_h(-1)^{h-k} \binom{h}{k}$; but this is just the (k, h) entry in the coefficient matrix of (8). We have demonstrated then that

$$\frac{t}{k} (Pw_{1,k}^+)'(t) = v_k(t).$$

Since v_k is nonnegative it follows that P preserves monotonicity.

We now complete the inductive step of the proof. With Theorem 2.1 in mind we define, for $1 \leqslant j \leqslant k \leqslant n$, $w_{j,k}(t) := (t - t_{k-j+1})(t - t_{k-j+2}) \cdots (t - t_{k-1})$ and

$$w_{j,k}^+(t) := \begin{cases} 0 & \text{if } \ 0 \leqslant t \leqslant t_{k-1} \\ w_{j,k}(t) & \text{if } \ t_{k-1} \leqslant t \leqslant 1, \end{cases}$$

where $t_i = i/n$. For $j \geqslant 1$ fixed, we claim that

$$\kappa t (Pw_{j+1,k}^+)^{(j+1)}(t) = (Pw_{j,k}^+)^{(j)}(t) \tag{9}$$

for $k=j+1, j+2, ..., n$, where κ is a nonnegative constant (recall that we have already seen $(Pw_{1,k}^{+})^{(1)}(t) \geqslant 0$ for every $k=1, ..., n$). To show (9), we begin by noting that, from the definition of $\omega_{j,k}$, it is clear that (with $k \geqslant j+1$) the degree of every term in polynomials $Pw_{j,k}^{+}$ and $Pw_{j+1,k}^{+}$ is greater than or equal to k. Indeed, for $k \leqslant h \leqslant n$, a careful calculation shows that the coefficient of t^h in $Pw_{j,k}^{+}$ is

$$C_{j,h,k} := a_h \frac{1}{n^{j-1}} \sum_{i=k}^{h} (-1)^{h-i} \binom{h}{i} \prod_{m=1}^{j-1} (i-k+m) \tag{10}$$

while the coefficient of t^h in $Pw_{j+1,k}^{+}$ is

$$C_{j+1,h,k} := a_h \frac{1}{n^j} \sum_{i=k}^{h} (-1)^{h-i} \binom{h}{i} \prod_{m=1}^{j} (i-k+m). \tag{11}$$

Using Lemma 4.1, it is possible to obtain $C_{j+1,h,k}$ from $C_{j,h,k}$ via multiplication by a constant (dependent on h),

$$\begin{aligned}
\frac{n(h-j)}{j(k-j)} C_{j+1,h,k} &= \frac{n(h-j)}{j(k-j)} a_j \frac{1}{n^j} \sum_{i=k}^{h} (-1)^{h-i} \binom{h}{i} \prod_{m=1}^{j} (i-k+m) \\
&= a_h \frac{1}{n^{j-1}} \left(\frac{h-j}{j(k-j)} (-1)^{h-k} h! \binom{h-j-1}{k-j-1} \right) \\
&\qquad \text{by Lemma 4.1} \\
&= a_h \frac{1}{n^{j-1}} (-1)^{h-k} (j-1)! \binom{h-j}{k-j} \\
&= C_{j,h,k} \qquad \text{by Lemma 4.1.}
\end{aligned}$$

Let $\kappa := n/j(k-j)$; then the coefficient of t^{h-j} in the polynomial $\kappa t(P\omega_{j+1,k}^{+})^{(j+1)}(t)$ is

$$\begin{aligned}
&\frac{n}{j(k-j)} C_{j+1,h,k} h(h-1) \cdots (h-j+1)(h-j) \\
&\quad = C_{j,h,k} h(h-1) \cdots (h-j+1).
\end{aligned} \tag{12}$$

But the right-hand side of (12) is simply the coefficient of t^{h-j} in the polynomial $(P\omega_{j,k}^{+})^{(j)}(t)$ and thus we have established (9). Therefore, $Pw_{j,k}^{+}(t)$ is j-convex for $j=1, ..., n$ $(k=j, ..., n)$ and thus, by Theorem 2.1, P preserves j-convexity for $j=1, ..., n$. ∎

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